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Partial differential equations from matrices with orthogonal columns

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Abstract. We discuss a system of third order PDEs for strictly convex smooth functions on domains of Euclidean space. We argue that it may be understood as the closure of the first order prolongation of a family of PDEs. We describe explicitly its real analytic solutions and all the solutions which satisfy a genericity condition; we also describe a family of non-generic solutions which has an application to Poisson geometry and Kähler structures on toric varieties. Our methods are geometric: we use the theory of Hessian metrics and symmetric spaces to link the analysis of the system of PDEs with properties of the manifold of matrices with orthogonal columns.

1. Introduction

Let ϕ be a strictly convex smooth function defined on a connected subset $\Omega \subset \mathbb{R}^n$. Its Hessian $H\phi$ defines at each point an inner product, and therefore the inverse of the Hessian matrix is also a smooth field of inner products. It is natural to ask whether this field is also the Hessian of a function. If g is a field of inner products which is the Hessian of a function, then there must be an equality of partial derivatives:

$$\frac{\partial g_{ij}}{\partial x_k} = \frac{\partial g_{ik}}{\partial x_j}, \quad 1 \leq i, j, k \leq n.$$

If Ω has trivial first homology group, then the agreement of the above partial derivatives implies that g is the Hessian of a function [3]. We shall assume from now on that the domain Ω has trivial first homology group.

Definition 1.1. A strictly convex function $\phi \in C^\infty(\Omega)$ has *property* $\mathcal{I}$ if it satisfies the following system of third order PDEs:

$$\frac{\partial}{\partial x_k} H\phi^{-1}{}_{ij} - \frac{\partial}{\partial x_j} H\phi^{-1}{}_{ik} = 0, \quad 1 \leq i, j, k \leq n. \tag{1.1}$$

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The purpose of this paper is to analyze the system of third order PDEs (1.1) for strictly convex functions.

To be more precise about our focus, we note that it is possible to construct strictly convex solutions to (1.1) by elementary means: every strictly convex function of one variable has property $\mathscr{I}$. If $\phi_1(x_1)$ and $\phi_2(x_2)$ are strictly convex functions of one variable, then $\phi_1(x_1) + \phi_2(x_2)$ has property $\mathscr{I}$. Such a function solves the second order hyperbolic PDE with constant coefficients

$$\frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0, \tag{1.2}$$

and, conversely, all the solutions of (1.2) decompose (locally) as the sum of two functions on each of the variables x_1 and x_2; the parallel translates of the coordinate axes are the (constant) characteristics of the solutions. If (1.2) is replaced by any second order hyperbolic PDE with constant coefficients whose solutions have (constant) orthogonal characteristics, then its strictly convex solutions will have property $\mathscr{I}$. There is a natural generalization of this family of hyperbolic second order PDEs to arbitrary dimensions. Its strictly convex solutions, which we refer to as functions with (constant) orthogonal characteristics, will also have property $\mathscr{I}$.

It is thus natural to study 'how close' a strictly convex function with property $\mathscr{I}$ may be from having orthogonal characteristics.

Our main results describe sufficient conditions for a strictly convex function with property $\mathscr{I}$ to have orthogonal characteristics. Among such sufficient conditions, there is a generic one:

Theorem 1.2. *If a strictly convex function has property $\mathscr{I}$ and at every point the eigenvalues of its Hessian are simple, then it has orthogonal characteristics.*

Another sufficient condition refers to the regularity of the functions:

Theorem 1.3. *If a real analytic strictly convex function has property $\mathscr{I}$, then it has orthogonal characteristics.*

We shall prove Theorem 1.2 first for functions of two variables. For them, the system (1.1) has two equations and the theorem will follow from an algebraic manipulation valid under the hypothesis on the Hessian. The algebraic manipulation will have a geometric counterpart: the family of second order hyperbolic PDEs with constant coefficients whose solutions have orthogonal characteristics defines a pencil of hyperplanes on the space of jets of order two; away from its base, and in the set where the Hessian is strictly positive, it restricts to a foliation. Strictly convex functions with property $\mathscr{I}$ define a subset[1] of the space of jets of order three. The hypothesis on the eigenvalues of the Hessian singles out the locus of smooth points for which the jet projection is a submersion; its image is the aforementioned foliated open subset of the jets of order two. The geometric manifestation of our algebraic manipulation will be that the Cartan connection is tangent to the leaves of the pullback foliation. This is why one may say that, for strictly convex

[1] That strictly convex functions with orthogonal characteristics have property $\mathscr{I}$ means that this subset contains the prolongation of any of the previous hyperplanes.

functions, the system of third order PDEs (1.1) is the closure of the prolongation of the aforementioned pencil of second order hyperbolic PDEs.

To go to arbitrary dimensions, we will not follow the jet space approach, as we find the algebraic complexities difficult to manage. We shall switch our viewpoint to that of Hessian metrics. In this language, what we are asking is when, for a given Hessian metric on a domain of Euclidean space, its inverse metric is also Hessian. We refer to such metrics as Hessian metrics with property $\mathcal{I}$. The first manifestation of the relevance of the metric viewpoint will be the following.

Lemma 1.4. *A Hessian metric has property $\mathcal{I}$ if and only if its Christoffel symbols* (*of the second kind*) *are symmetric in the three indices.*

Lemma 1.4 suggests that property $\mathcal{I}$ could be described as a feature of the tangent or the orthogonal frame bundle of the Hessian metric with its Levi-Civita connection. A fundamental property of Hessian metrics on domains of Euclidean space is that they posses a universal (positively oriented) orthogonal frame bundle with connection [3], $\pi\colon (\mathrm{Gl}(n)^+, \nabla) \to \mathcal{P}$, where $\mathcal{P}$ denotes the set of positive matrices, and π and ∇ are a natural map and connection, respectively, which come from symmetric space theory. We shall argue that the universal orthogonal frame bundle offers an appropriate replacement for the jet space picture. Briefly, jet spaces of order two will be replaced by the set $\mathcal{P}$ of positive matrices; the subset of the jet spaces of order three defined by property $\mathcal{I}$ will be replaced by the submanifold of matrices with orthogonal columns $\mathcal{C} \subset \mathrm{Gl}(n)^+$; the restriction of the jet projection will correspond to $\pi|_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{P}$; and the Cartan connection will correspond to the universal Levi-Civita connection ∇. Property $\mathcal{I}$ for a Hessian metric will translate as follows:

Theorem 1.5. *A Hessian metric $H\phi$ in Ω has property $\mathcal{I}$ if and only if for any point $x \in \Omega$ and any curve γ at $x \in \Omega$, there exists an orthonormal frame for $H\phi(x)$ in $\mathcal{C}$ such that the corresponding horizontal lift of γ at that frame is tangent to $\mathcal{C}$.*

There will be a property analogous to the tangency of the Cartan connection to the pullback foliation coming from the pencil of degree two hyperbolic PDEs:

Proposition 1.6. *The restriction of the universal Levi-Civita connection to $\mathcal{C}$ defines a* (*regular*) *involutive distribution. Its leaves are the left translates of the strictly positive matrices $\mathcal{D}$ which fit in the Cartan factorization $\mathcal{C} = \mathrm{SO}(n)\,\mathcal{D}$.*

The generic condition on eigenvalues in Theorem 1.2 is just the open stratum of a natural stratification of $\mathcal{P}$. To describe more precisely sufficient conditions for a Hessian metric with property $\mathcal{I}$ to come from a function with orthogonal characteristics, we will analyze the interaction among this stratification, the foliation on $\mathcal{C}$ defined by the universal Levi-Civita connection, and the map $\pi|_{\mathcal{C}}$. Theorem 1.3 will hinge on real analytic features of these objects.

As we shall see, property $\mathcal{I}$ for strictly convex functions appears in a problem of Poisson geometry in toric varieties. The so-called totally real toric Poisson structures have properties analogous to that of Hamiltonian Kähler forms. For instance, whereas the latter are encoded by appropriate strictly convex functions [5], the former are encoded by the simplest strictly convex functions: quadratic forms. The most natural Poisson-theoretic

PDE for a pair given by a totally real toric Poisson structure and a Hamiltonian Kähler form will correspond to property $\mathcal{I}$:

Theorem 1.7. *Let* $(X, \mathbb{T})$ *be a* (*smooth*) *toric variety endowed with a totally real toric Poisson structure* Π *and a Kähler form* σ *for with the action of the maximal compact torus* $T \subset \mathbb{T}$ *is Hamiltonian. Let* P *denote the inverse Poisson structure to* σ. *Then the following statements are equivalent*:

(1) *The Poisson structures* Π *and* P *Poisson commute*: $[\Pi, P] = 0$.

(2) *In a basis of the Lie algebra of* T *for which* Π *corresponds to the standard quadratic from of* $\mathbb{R}^n$, *the strictly convex function which corresponds to* σ *has property* $\mathcal{I}$.

In complex dimension one, a totally real toric Poisson structure and (the inverse of) a Hamiltonian Kähler form always Poisson commute because the commutator is a field of trivectors on a surface; equivalently, if we use Theorem 1.7, this corresponds to the fact that all strictly convex functions of one variable have property $\mathcal{I}$. As it will turn out, Theorem 1.3 will imply that in the real analytic category, such a commuting pair is the Cartesian product of one dimensional commuting pairs:

Theorem 1.8. *Let* $(X, \mathbb{T})$ *be a projective toric Poisson variety endowed with a totally real toric Poisson structure* Π *which Poisson commutes with a real analytic Hamiltonian Kähler structure* σ. *Then* $(X, \mathbb{T})$ *is a Cartesian product of projective lines, and both* Π *and* σ *factorize.*

We shall also describe a family of strictly convex functions which satisfy property $\mathcal{I}$ but which do not have orthogonal characteristics. We will use it to construct Hamiltonian Kähler forms in certain (T-invariant) regions of toric varieties. These regions can be thought of as the result of gluing to a (T-round) 0-handle several (T-round) 1-handles. An illustration of the construction is the following:

Proposition 1.9. *Let* $U \subset \mathbb{C}P^2$ *be the complement of small* T^2*-invariant neighborhoods of* $[1 : 0 : 0]$ *and* $[1 : 0 : 1]$. *Then there exist a totally real toric Poisson structure on* $\mathbb{C}P^2$ *and a Hamiltonian Kähler form on* U *which Poisson commute.*

The structure of this paper is as follows. Section 2 describes how matrices with orthogonal columns are used to define the family of differential relations of second order with constant coefficients whose solutions we call functions with (constant) orthogonal characteristics; we also discuss why they have property $\mathcal{I}$. In Section 3, we do the analysis of the system of third order PDEs (1.1) for strictly convex functions of two variables using jet spaces. The viewpoint of Hessian metrics is introduced in Section 4. Property $\mathcal{I}$ is translated as symmetry of the Christoffel symbols, an algebraically simpler condition which allows to analyze the interaction of property $\mathcal{I}$ with the Legendre transform. Section 5 describes how the universal orthogonal frame bundle offers the appropriate setting for the geometric analysis of property $\mathcal{I}$. We analyze the map $\pi\colon (\mathcal{C}, \nabla) \to \mathcal{P}$ from the submanifold of orthogonal matrices with the restriction of the universal Levi-Civita connection onto the manifold of positive matrices; this is our replacement of the subsets defined by property $\mathcal{I}$ in the space of jets of order three and two with the restriction of the Cartan connection. Section 6 contains our main results: firstly, the description of property $\mathcal{I}$ as a differential relation related to the submanifold of orthogonal matrices $\mathcal{C} \subset (\mathrm{Gl}(n)^+, \nabla) \to \mathcal{P}$. Secondly, sufficient conditions for a Hessian metric with property $\mathcal{I}$ to come from a strictly convex function with orthogonal characteristics. In Section 7, we describe a family of strictly convex functions which have property $\mathcal{I}$ but do not have in general orthogonal characteristics. The domains of definition of its members are what we call polytopes with 1-handles. Despite polytopes with 1-handles are not convex in general, we show that the family is invariant under Legendre transform. Section 8 contains our applications to Poisson geometry. We explain how on a smooth toric variety, the Poisson commuting equation for a totally real toric Poisson structure and for (the inverse of) a Hamiltonian Kähler form can be rewritten as property $\mathcal{I}$ for either the Kähler or the symplectic potential [5] of the latter form. That allows us to conclude that in the real analytic category, any such commuting pair must be the Cartesian product of commuting pairs on projective lines. We also use the family introduced in Section 7 to construct commuting pairs on certain topologically non-trivial regions of toric varieties (which are not Cartesian products).

2. Solutions with orthogonal characteristics

It is possible to construct strictly convex functions with property $\mathcal{I}$ by elementary means.

(a) Every strictly convex function of one variable has property $\mathcal{I}$.

(b) If $\phi_1(x_1), \dots, \phi_n(x_n)$ are strictly convex, then $\phi_1(x_1) + \cdots + \phi_n(x_n)$ has property $\mathcal{I}$ in the product of the corresponding intervals.

(c) If $\phi(x)$ has property $\mathcal{I}$ in Ω and $B \in \mathrm{O}(n)$ is an orthogonal transformation, then $\phi(Bx)$ has property $\mathcal{I}$ in $B^{-1}(\Omega)$. This is because

$$H\phi(Bx) = B^{\top} H\phi(x) B$$

and, therefore, if $H\phi(x)^{-1}$ is the Hessian of $\psi(x)$, then $(H\phi(Bx))^{-1}$ is the Hessian of $\psi(Bx)$.

A function is (locally) of the form $\phi = \phi_1(x_1) + \cdots + \phi_n(x_n)$ if and only if it is a solution of the system of second order PDEs

$$\frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0, \quad 1 \leq i < j \leq n. \tag{2.1}$$

The solutions of the system have (constant) characteristics given by the collection of axes. This information can be used to rewrite (2.1) in a more geometric fashion. For any $n \times n$ matrix A, one can define a differential operator of order two on functions with values on matrix-valued functions by the following recipe:

$$\mathcal{L}_A^2 \phi := A^{\top} H\phi A. \tag{2.2}$$

Equivalently, the ij-th component is the Lie derivative of ϕ with respect to the (constant) vector field defined by the i-th column of A, followed by the Lie derivative with respect to the vector field defined by the j-th column.

Let $\mathcal{D}$ denote the set of diagonal matrices with strictly positive entries, and let $\mathcal{D}(\Omega)$ denote the set of smooth functions on Ω with values on $\mathcal{D}$. It follows that a function ϕ

is strictly convex and satisfies (2.1) if and only if $\mathcal{L}^2_{\mathrm{I}}\phi \in \mathcal{D}(\Omega)$, where I is the identity matrix. Let $\mathcal{C}$ denote the set of matrices with orthogonal columns.

Definition 2.1. A function $\phi \in C^\infty(\Omega)$ has *orthogonal characteristics* if there exists $C \in \mathcal{C}$ such that

$$\mathcal{L}^2_C\phi \in \mathcal{D}(\Omega). \tag{2.3}$$

Because $\mathcal{D}$ is invariant by conjugation by permutation matrices, in Definition 2.1 we may assume that C has positive determinant. We will abuse notation and use $\mathcal{C}$ to refer to matrices with orthogonal columns and positive determinant.

Lemma 2.2. *If $\phi \in C^\infty(\Omega)$ has orthogonal characteristics, then ϕ is an strictly convex function with property $\mathcal{I}$. More precisely, ϕ is the composition of an orthogonal transformation with a function with trivial mixed partial derivatives.*

Proof. Because C has orthogonal columns, we can factor $C = B\Lambda$, with $\Lambda \in \mathcal{D}$ and $B \in \mathrm{SO}(n)$. From $\mathcal{L}^2_C\phi = C^\top H\phi C = \Lambda B^\top H\phi B\Lambda$ and (2.3), we deduce that $B^\top H\phi B \in \mathcal{D}(\Omega)$, or, equivalently, that $H\phi(Bx) \in \mathcal{D}(\Omega)$. Thus, locally,

$$\phi(Bx) = \phi_1(x_1) + \cdots + \phi_n(x_n), \quad x = (x_1, \ldots, x_n) \in B^{-1}(\Omega),$$

and ϕ_i is strictly convex. Therefore $\phi(Bx)$ is strictly convex and has property $\mathcal{I}$, and so the same occurs for $\phi(x) = \phi(B^{-1}(Bx))$. ∎

3. The two-dimensional case

We would like to know whether there exist strictly convex functions with property $\mathcal{I}$ which do not have orthogonal characteristics. For that we find convenient to discuss the algebraic structure of the system of PDEs (1.1). This should be easier in the lowest non-trivial dimension.

We shall denote partial derivatives of a function $\phi(x)$, $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$, by means of subindices which follow a comma. We introduce independent variables to parametrize (homogeneous) jet spaces of order two and three:

$$\chi = \phi_{,11}, \quad \tau = \phi_{,12}, \quad \zeta = \phi_{,22}, \quad \upsilon = \phi_{,111}, \quad \nu = \phi_{,112}, \quad \omega = \phi_{,122}, \quad \xi = \phi_{,222}.$$

Strictly convex functions correspond to the open subset $\chi\zeta - \tau^2 > 0$, $\chi + \zeta > 0$. The system of PDEs (1.1) corresponds to the common solutions of the following degree three homogeneous polynomial equations:

$$\begin{cases} \xi(\chi\zeta - \tau^2) - \zeta(\nu\zeta + \chi\xi - 2\tau\omega) + \nu(\chi\zeta - \tau^2) - \tau(\upsilon\zeta + \chi\omega - 2\tau\nu) = 0, \\ -\omega(\chi\zeta - \tau^2) + \tau(\nu\zeta + \chi\xi - 2\tau\omega) - \upsilon(\chi\zeta - \tau^2) + \chi(\upsilon\zeta + \chi\omega - 2\tau\nu) = 0. \end{cases} \tag{3.1}$$

Lemma 3.1. *Strictly convex functions with property $\mathcal{I}$ correspond, in the space of jets of order three of functions in the plane, to an open subset of an intersection of quadrics*:

$$\begin{cases} (\zeta - \chi)\nu + \tau(\upsilon - \omega) = 0, \\ (\zeta - \chi)\omega + \tau(\nu - \xi) = 0, \end{cases} \qquad \chi\zeta - \tau^2 > 0, \ \chi + \zeta > 0. \tag{3.2}$$

Proof. We interpret the equations of the cubics (3.1) as a (non-homogeneous) linear system with indeterminates $\nu\zeta + \chi\xi - 2\tau\omega$ and $\upsilon\zeta + \chi\omega - 2\tau\nu$:

$$\begin{cases} \zeta(\nu\zeta + \chi\xi - 2\tau\omega) + \tau(\upsilon\zeta + \chi\omega - 2\tau\nu) = (\xi + \nu)(\chi\zeta - \tau^2), \\ \tau(\nu\zeta + \chi\xi - 2\tau\omega) + \chi(\upsilon\zeta + \chi\omega - 2\tau\nu) = (\upsilon + \omega)(\chi\zeta - \tau^2). \end{cases}$$

In the open subset defined by $\chi\zeta - \tau^2 \neq 0$, we obtain the equivalent relations

$$\begin{cases} \nu\zeta + \chi\xi - 2\tau\omega = \begin{vmatrix} \xi + \nu & \tau \\ \upsilon + \omega & \chi \end{vmatrix}, \\ \upsilon\zeta + \chi\omega - 2\tau\nu = \begin{vmatrix} \zeta & \xi + \nu \\ \tau & \upsilon + \omega \end{vmatrix}, \end{cases} \iff \begin{cases} (\zeta - \chi)\nu + \tau(\upsilon - \omega) = 0, \\ (\zeta - \chi)\omega + \tau(\nu - \xi) = 0. \end{cases} \qquad \blacksquare$$

To each $[a:b] \in \mathbb{R}P^1$, one can associate the following second order PDE with constant coefficients for strictly convex functions:

$$a\phi_{,11} - a\phi_{,22} = b\phi_{,12}. \tag{3.3}$$

It corresponds to an open subset of a hyperplane of the space of jets of order two:

$$a(\chi - \zeta) - b\tau = 0, \quad \chi\zeta - \tau^2 > 0,\ \chi + \zeta > 0, \tag{3.4}$$

whose first prolongation is

$$\begin{cases} a(\chi - \zeta) - b\tau \ = 0, \\ a(\upsilon - \omega) - b\nu = 0, \\ a(\nu - \xi) - b\omega \ = 0, \end{cases} \quad \chi\zeta - \tau^2 > 0,\ \chi + \zeta > 0. \tag{3.5}$$

Proposition 3.2. *Let $\phi \in C^\infty(\Omega)$ be a strictly convex function.*

(1) *If ϕ satisfies (3.3) for some $[a:b] \in \mathbb{R}P^1$, then ϕ has property $\mathcal{I}$.*
(2) *If ϕ satisfies (3.3) for more than one $[a:b] \in \mathbb{R}P^1$, then ϕ is – up to a degree one polynomial – a multiple of the standard quadratic form $x_1^2 + x_2^2$.*
(3) *If ϕ has property $\mathcal{I}$ and its Hessian has simple eigenvalues, then ϕ satisfies (3.3) for some $[a:b] \in \mathbb{R}P^1$.*

Proof. The set of equations (3.3) are exactly those second order PDEs whose solutions have orthogonal characteristics. Therefore item (1) is the specialization of Lemma 2.2 to the two-dimensional case. Alternatively, item (1) follows from the inclusion of the solutions of (3.5) in the solutions of (3.2).

If ϕ satisfies (3.3) for more than one $[a:b] \in \mathbb{R}P^1$, then its second jet belongs to the base of the pencil (3.4): $\chi - \zeta = \tau = 0$. That is to say $\phi_{,12} = 0$ and $\phi_{,11} = \phi_{,22}$. Therefore $0 = \phi_{,221} = \phi_{,111} = \phi_{,112} = \phi_{,222}$. Hence ϕ is a degree two polynomial whose homogeneous part of degree two equals $k(x_1^2 + x_2^2)$, $k > 0$.

The Hessian $H\phi$ has two eigenvalues if and only if it misses the base of the pencil. Equivalently, the field of vectors in the plane $(\phi_{,22} - \phi_{11}, \phi_{,12}) \in \mathbb{R}^2$ has no zeroes.

Therefore we can (locally) take the quotient of the components of the vector field to get a well-defined slope function. Property $\mathcal{I}$ as in (3.2) can be rewritten

$$\begin{cases} \langle(\phi_{,22} - \phi_{11}, \phi_{,12}), (\phi_{12}, \phi_{,11} - \phi_{22})_{,1}\rangle = 0, \\ \langle(\phi_{,22} - \phi_{11}, \phi_{,12}), (\phi_{12}, \phi_{,11} - \phi_{22})_{,2}\rangle = 0, \end{cases}$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product. This implies that the slope function is constant, which is exactly the second order PDE (3.3) for some $[a:b] \in \mathbb{R}P^1$. ∎

Proposition 3.2 does not clarify whether strictly convex functions with property $\mathcal{I}$ and which do not have orthogonal characteristics exist. As we shall discuss in Section 7, such solutions exist: it is possible to start from a multiple of the standard quadratic form in a subdomain of Ω which 'bifurcates' into solutions to different equations in (3.3) in other subsets of the domain Ω.

Remark 3.3 (The Cartan connection on jet spaces). The algebraic manipulation in item (3) in Proposition 3.2 has a geometric counter-part. The requirement on the Hessian corresponds to the regularity condition needed to identify solutions with holonomic sections with respect to the Cartan connection: on the one hand, the subset of the jet spaces which corresponds to property $\mathcal{I}$ is not smooth; the 1-forms

$$\Xi_1 = \nu(d\zeta - d\chi) + (\zeta - \chi)d\nu + \tau(d\upsilon - d\omega) + (\upsilon - \omega)\,d\tau,$$
$$\Xi_2 = \omega(d\zeta - d\xi) + (\zeta - \xi)\,d\omega + \tau(d\nu - d\varepsilon) + (\nu - \varepsilon)\,d\tau,$$

are colinear in the subset $\omega(\upsilon - \omega) - \nu(\nu - \varepsilon) = \chi - \zeta = \tau = 0$. On the other hand, the smooth locus of the intersection of quadrics fails to be transverse to the fibers of the projection onto jets of order two in the points over the base of the pencil.

The connection 1-forms one has to add when passing from jets of order two to jets of order three are:

$$\Theta_1 = d\xi - \upsilon\,dx - \nu dy, \quad \Theta_2 = d\tau - \nu\,dx - \omega\,dy \quad \text{and} \quad \Theta_3 = d\chi - \omega\,dx - \zeta dy.$$

The pullback foliation is defined by the 1-forms

$$K_1 = (\xi - \chi)\,d\theta - \tau(d\xi - d\chi), \quad K_2 = (\upsilon - \omega)\,d\nu - \nu(d\upsilon - d\omega),$$
$$K_3 = (\nu - \varepsilon)\,d\omega - \omega(d\nu - d\varepsilon).$$

The equalities

$$K_1 = (\chi - \zeta)\,\Theta_2 - \tau(\Theta_1 - \Theta_3), \quad K_2 - \frac{\nu^2}{\tau^2}K_1 = -\frac{\nu}{\tau}\,\Xi_1, \quad K_3 - \frac{\omega^2}{\tau^2}K_1 = -\frac{\omega}{\tau}\,\Xi_2$$

hold in the intersection of (3.2) with the complement of the pullback of the base of the pencil. Therefore, holonomic sections in this subset are tangent to the pullback foliation. Hence their order two jet must be inside a hyperplane of the pencil.

Remark 3.4. (Orthogonal characteristics and Legendre transform) Let ϕ be an strictly convex function on a convex domain $\Omega \subset \mathbb{R}^2$. Its Legendre transform is an strictly convex function ϕ^* on a convex domain Ω^*, which is related to Ω by a (Legendre) diffeomorphism. Let us assume that ϕ satisfies the constant coefficients second order PDE:

$$a\phi_{,11} + c\phi_{,22} - b\phi_{,12} = 0, \quad [a:c:b] \in \mathbb{R}P^2.$$

Because $H\phi^*$ at $x \in \Omega^*$ equals $H\phi^{-1}$ at its related point in Ω, we have the equality

$$a\phi^*_{,22} + c\phi^*_{,11} + b\phi^*_{,12} = 0.$$

Thus the Legendre transform induces an involution in the parameter space of constant coefficients degree two homogeneous PDEs: $[a:c:b] \mapsto [c:a:-b]$. Its fixed point set is $[1:1:0] \cup [a:-a:b] \subset \mathbb{R}P^2$. To the point $[1:1:0]$ corresponds the Laplace equation, which has no strictly convex solutions. The projective line $[a:-a:b]$ parametrizes hyperbolic PDEs with orthogonal characteristics (3.3). Therefore, if ϕ is a function on the convex domain Ω with orthogonal characteristics, so its Legendre transform is. This invariance property holds regardless of the dimension:

$$\mathcal{L}^2_C\phi \in \mathcal{D}(\Omega) \iff C^\top (H\phi^*)^{-1} C \in D(\Omega^*) \iff C^\top (H\phi^*) C \in \mathcal{D}(\Omega^*),$$

where the first equivalence uses the relation between Hessian matrices of the original function and its Legendre transform, and in the second equivalence we have inverted the matrices and we have used $C^\top C \in \mathcal{D}$.

4. Hessian metrics with symmetric Christoffel symbols

To generalize the results in Section 3, the complexities brought by the increase of dimension shall be dealt with by shifting the perspective to that of Hessian metrics.

A *Hessian metric* on $\Omega \subset \mathbb{R}^n$ is a Riemannian metric obtained as the Hessian matrix of a (strictly convex) function on Ω. Property $\mathcal{I}$ for strictly convex functions can be translated to a requirement for a Hessian metric: that its inverse metric be Hessian as well. In such case, we say that the given Hessian metric has property $\mathcal{I}$.

There is another natural differential condition on Hessian metrics which allows to formulate in arbitrary dimensions the algebraic simplification of property $\mathcal{I}$ described in Lemma 3.1. For a Hessian metric $H\phi$, the Christoffel symbols of the first kind equal the partial derivatives of order three: $\Gamma_{ijk} = \phi_{,ijk}$. The Christoffel symbols (of the second kind) are

$$\Gamma^i_{kj} = H\phi^{-1}{}_{il}\,\Gamma_{ljk}.$$

Let $[H\phi]_{,k}$ denote the partial derivative with respect to k of the entries of the Hessian matrix. For each $1 \le k \le n$, we define the *Christoffel matrix*

$$\Gamma_k := \Gamma^\bullet_{k\star} = H\phi^{-1}{}_{\bullet\circ}[H\phi]_{,k\,\circ\star} = H\phi^{-1}[H\phi]_{,k}.$$

Here the symbols $\bullet$ and $\star$ denote the superindex and second subindex for Christoffel symbols of the first kind, and also the row and column indices in the matrices $H\phi^{-1}$ and $[H\phi]_{,k}$, respectively; the symbol $\circ$ denotes the row index in the matrix $[H\phi]_{,k}$.

Definition 4.1. A Hessian metric $H\phi$ on Ω has *symmetric Christoffel symbols* if the Christoffel symbols (of the second type) are symmetric on the three indices. Equivalently, if its Christoffel matrices are symmetric.

Property $\mathcal{I}$ corresponds to an open subset of the solutions of a system of polynomial equations of degree $2n-1$ in the space of jets of order three. The symmetry of the Christoffel symbols is determined by a system of polynomial equations of degree n; for $n=2$, it is exactly (3.2). The generalization of Lemma 3.1 to arbitrary dimensions is that property $\mathcal{I}$ translates into the symmetry of Christoffel symbols:

Proof of Lemma 1.4. The Hessian metric is invertible if and only if the i-th and j-th lines of $[H\phi^{-1}]_{,j}$ and $[H\phi^{-1}]_{,i}$ are equal. This is equivalent to the same condition for the matrices $[H\phi^{-1}]_{,j} H\phi$ and $[H\phi^{-1}]_{,i} H\phi$. If we prolong the identity $H\phi^{-1} H\phi = I$, then the condition transforms into the same condition for the Christoffel matrices Γ_j and Γ_i. This amounts to symmetry of all Christoffel matrices. ■

The problem of the symmetry of Christoffel symbols of Hessian metrics is amenable to Lie theoretic methods. A first instance of that is the following.

Proposition 4.2. *The following statements for a Hessian metric $H\phi$ on Ω are equivalent*:

(1) *It has symmetric Christoffel symbols.*

(2) *There exists a Cartan subalgebra* (*of the Lie algebra of $n \times n$ matrices*) *inside of the symmetric matrices to which the two matrices $H\phi$ and $[H\phi]_{,k}$ belong, for $1 \leq k \leq n$.* (*The Cartan subalgebra may vary with k.*)

(3) *There exists a Cartan subalgebra inside of the symmetric matrices to which the two matrices $H\phi$ and Γ_k belong, for $1 \leq k \leq n$.* (*The Cartan subalgebra may vary with k.*)

Proof. Let $\mathfrak{s}$ be the vector subspace of symmetric matrices, and let $\mathfrak{d} \subset \mathfrak{s}$ denote the diagonal matrices; this is a Cartan subalgebra of the Lie algebra of $n \times n$ matrices.

The Christoffel matrix Γ_k is the product of the symmetric matrices $H\phi^{-1}$ and $[H\phi]_{,k}$. Therefore $H\phi$ has symmetric Christoffel matrices if and only if the following commutators are trivial:

$$[H\phi^{-1}, [H\phi]_{,k}] = 0, \quad 1 \leq k \leq n.$$

This is equivalent to require that $[H\phi]_{,k}$ be in the same Cartan subalgebra as $H\phi^{-1}$, and, because both are symmetric matrices, this Cartan subalgebra must lie in $\mathfrak{s}$. If B is a orthogonal matrix which diagonalizes $H\phi$, then it also diagonalizes $H\phi^{-1}$:

$$B^{\top} H\phi B = \Lambda, \quad B^{\top} H\phi^{-1} B = \Lambda^{-1}.$$

Therefore, if $H\phi, [H\phi]_{,k}$ are in the Cartan subalgebra $\mathrm{Ad}_B(\mathfrak{d}) \subset \mathfrak{s}$, then so is $H\phi^{-1}$. This shows the equivalence between (1) and (2).

If (2) holds, then

$$\Gamma_k = H\phi^{-1}[H\phi]_{,k} = B^{\top}\Lambda_1 B B^{\top}\Lambda_2 B = B^{\top}\Lambda_1\Lambda_2 B$$

remains in the same Cartan subalgebra of the commuting factors, which proves (3). Condition (3) by definition implies the symmetry of the Christoffel matrices. ■

By Lemma 1.4, Hessian metrics with symmetric Christoffel symbols are the same as Hessian metrics with property $\mathcal{I}$. Thus, by Lemma 2.2, strictly convex functions with orthogonal characteristics define Hessian metrics with symmetric Christoffel symbols. We can reprove this result with a Lie theoretic approach:

Lemma 4.3. *Let $\phi \in C^\infty(\Omega)$. If $\mathcal{L}^2_C\phi \in \mathcal{D}(\Omega)$ for some $C \in \mathcal{C}$, then the Christoffel matrices of $H\phi$ for all points in Ω are in the Cartan subalgebra* $\mathrm{Ad}_C(\mathfrak{d})$. *In particular, $H\phi$ has symmetric Christoffel symbols.*

Proof. By the hypotheses, for each $x \in \Omega$,

$$C^\top H\phi C = \Lambda, \quad \Lambda = \Lambda(x) \in \mathcal{D}(\Omega), \quad C \in \mathcal{C}.$$

Hence

$$H\phi = (C^\top)^{-1}\Lambda C^{-1},$$

and upon taking its first order prolongation,

$$[H\phi]_{,k} = (C^\top)^{-1}\Lambda_{,k}C^{-1}.$$

Therefore both $H\phi$ and $[H\phi]_{,k}$ are in $\mathrm{Ad}_C(\mathfrak{d})$. By item (2) in Proposition 4.2 the same occurs for Γ_k (the action by conjugation on $\mathfrak{d}$ of second factor of $\mathcal{C} = \mathrm{SO}(n)\mathcal{D}$ is trivial). By Proposition 4.2, this implies the symmetry of Christoffel matrices. ∎

Proposition 4.4. *The Legendre transform preserves the class of Hessian metrics with property $\mathcal{I}$ on convex domains.*

Proof. Let $\mathcal{L}_j$ denote the Lie derivative with respect to $\partial/\partial x_j$. We can rewrite property $\mathcal{I}$ for $H\phi$ as

$$\mathcal{L}_\bullet(H\phi^{-1})_{\star k} - \mathcal{L}_\star(H\phi^{-1})_{\bullet k} = 0. \tag{4.1}$$

The differential of ϕ defines the Legendre diffeomorphism

$$d\phi : \Omega \to \Omega^*, \quad D(d\phi) = H\phi.$$

If we push forward each equation in (4.1) by the Legendre diffeomorphism $d\phi$, then the Lie derivative of the pushed forward functions – entries of the inverse Hessian – by the pushed forward vector fields will subtract to zero as well. The entries of the inverse Hessian matrix are pushed forward to the entries of the Hessian of ϕ^*; the coordinate vector fields are pushed forward to the columns vector fields of the Jacobian matrix $H\phi$, which at points in Ω^* is the matrix $H\phi^{*-1}$. Therefore, property $\mathcal{I}$ for $H\phi$ is equivalent to

$$H\phi^{*-1}{}_{\bullet\circ}[H\phi^*]_{k\circ\star} - H\phi^{*-1}{}_{\star\circ}[H\phi^*]_{k\circ\bullet} = 0,$$

which is the symmetry of the Christoffel matrices of $H\phi^*$. Therefore, by Lemma 1.4, $H\phi^*$ has property $\mathcal{I}$. ∎

5. The universal frame bundle for Hessian metrics and matrices with orthogonal columns

To generalize Proposition 3.2 to arbitrary dimensions, jet spaces will be replaced by (a subset of) the principal orthogonal frame bundle of the Hessian metric with its Levi-Civita connection. There are three reasons to do that:

(a) A function $\phi \in C^\infty(\Omega)$ has property $\mathcal{I}$ if and only if $H\phi$ has symmetric Christoffel symbols. The Christoffel symbols are the components of the Levi-Civita connection. Therefore one may expect a reformulation of property $\mathcal{I}$ related to the tangent or orthogonal frame bundle with the Levi-Civita connection.

(b) If a function $\phi \in C^\infty(\Omega)$ has orthogonal characteristics, then the Hessian metric $H\phi$ splits (locally, but along the same characteristics everywhere). In other words, the conclusion of the de Rham splitting theorem holds. Therefore, to study the relation between property $\mathcal{I}$ and orthogonal characteristics, it may be appropriate to look at parallel transport on the principal frame bundle with its the Levi-Civita connection.

(c) Hessian metric on domains of Euclidean space are characterized among Riemannian metrics as those whose frame bundle is the pullback of a universal principal bundle with connection coming from symmetric space theory [3].

For a function ϕ, the information of the homogeneous part of its second jet is the same as the one contained in its Hessian. Thus for our strictly convex functions we shall be looking at the map $x \mapsto H\phi(x)$, which takes values in the positive matrices $\mathcal{P}$. There, the pencil in (3.4) defined by hyperbolic PDEs with orthogonal characteristics generalizes as follows: the second order PDE equation (2.1) corresponds to Hessian metrics with image in the (positive) diagonal matrices $\mathcal{D}$. Matrices with orthogonal columns have a factorization into an orthogonal and a diagonal matrix. Thus we may confine ourselves to the family of second order PDEs $\mathcal{L}^2_B\phi \in \mathcal{D}$, $B \in \mathrm{SO}(n)$. To each of them, there corresponds the subset $\mathrm{Ad}_B(\mathcal{D}) \subset \mathcal{P}$; their union over $B \in \mathrm{SO}(n)$ fills $\mathcal{P}$, as any positive matrix can be diagonalised by a special orthogonal transformation.

For a Riemannian metric defined on a subset of Euclidean space, its orthogonal frame bundle – forgetting for the moment about the Levi-Civita connection – is constructed via pullback: the map $\pi\colon \mathrm{Gl}(n)^+ \to \mathrm{Gl}(n)^+$, $A \mapsto {A^{-1}}^{\mathsf{T}} A^{-1}$, has as image the closed embedded submanifold of positive matrices. The restriction to its image,

$$\pi : \mathrm{Gl}(n)^+ \to \mathcal{P}, \tag{5.1}$$

- is a (right) principal bundle for $\mathrm{SO}(n)$;
- intertwines the right action of $\mathrm{SO}(n)$ on $\mathrm{Gl}(n)^+$ and the adjoint action of $\mathrm{SO}(n)$ on $\mathcal{P}$;
- is the bundle of (positively oriented) orthogonal frames for metrics on $\mathbb{R}^n$.

Let ∇ be the $\mathrm{SO}(n)$-invariant principal connection on $\pi\colon \mathrm{Gl}(n)^+ \to \mathcal{P}$ which at the identity matrix has as horizontal space the symmetric matrices[2].

[2] Its curvature there is $[\mathfrak{s}, \mathfrak{s}]$.

Proposition 5.1 (Proposition 4.1 in [3]). *If $H\phi$ is a Hessian metric on Ω, then the pull-back of ∇ by $H\phi\colon \Omega \to \mathcal{P}$, $x \mapsto H\phi(x)$, is the Levi-Civita connection on the orthogonal frame bundle of $H\phi$. Furthermore, this property characterizes Hessian metrics among Riemmanian metrics in domains of Euclidean space.*

The appropriate replacement of the jets of order three will not be the full bundle of orthogonal frames. It will be the subset of matrices with orthogonal columns. The following result, from which Proposition 1.6 in the introduction follows, shows that it is well-behaved with respect to the universal Levi-Civita connection:

Proposition 5.2. *The subset of matrices with orthogonal columns $\mathcal{C} \subset \mathrm{Gl}(n)^+$ has the following properties*:

(1) *It is a closed embedded submanifold of $\mathrm{Gl}(n)^+$ on which the Cartan decomposition $\mathrm{Gl}(n)^+ = \mathrm{SO}(n)\mathcal{P}$ induces a product structure $\mathcal{C} = \mathrm{SO}(n)\mathcal{D}$.*

(2) *The intersection of the horizontal distribution of ∇ with the tangent bundle $T\mathcal{C}$ is an involutive distribution on $\mathcal{C}$. Its foliation $\mathcal{F}$ is the one associated to the Cartan decomposition, with leaves the left $\mathrm{SO}(n)$-translates of $\mathcal{D}$.*

Proof. Let ι be the inversion map on $\mathrm{Gl}(n)^+$ and let $q = \pi \circ \iota : \mathrm{Gl}(n)^+ \to \mathcal{P}$, $A \mapsto A^\top A$. A matrix C has orthogonal columns if and only if $q(C) \in \mathcal{D}$. Therefore $\mathcal{C}$ is the preimage under a submersion of the closed embedded submanifold of positive diagonal matrices, thus a closed embedded submanifold of $\mathrm{Gl}(n)^+$. We have already used the (unique) factorisation of a matrix with orthogonal columns as a product of an orthogonal and a diagonal matrix. It is straightforward that it gives rise to a Cartesian product of manifolds $\mathcal{C} = \mathrm{SO}(n)\mathcal{D}$.

The product structure in (1) implies that its tangent space at $C \in \mathcal{C}$ is

$$\mathfrak{so}(n) \cdot C \cdot \mathfrak{d} = C \cdot \mathrm{Ad}_{C^{-1}}(\mathfrak{so}(n)) \cdot \mathfrak{d}.$$

The horizontal space of ∇ there is $C \cdot \mathfrak{s}$. Because the conjugation of a skew orthogonal matrix by an orthogonal one can never be symmetric, the intersection of the tangent spaces must be $C \cdot \mathfrak{d}$. Therefore, the intersection of the horizontal space of ∇ with $T\mathcal{C}$ is the distribution[3] tangent to the left translates of $\mathcal{D}$ by $\mathrm{SO}(n)$. ∎

Next, we argue how $\pi\colon (\mathcal{C}, \mathcal{F}) \to \mathcal{P}$ provides a 'desingularization' of the pencil $\mathrm{Ad}_B(\mathcal{D})$, $B \in \mathrm{SO}(n)$.

Proposition 5.3. *The restriction $\pi|_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{P}$ has the following properties*:

(1) *It is a surjective map all whose values are clean.*

(2) *The restriction of the differential of $\pi|_{\mathcal{C}}$ to $T\mathcal{F}$ has trivial kernel, and the restriction of $\pi|_{\mathcal{C}}$ to the leaf $B\mathcal{D}$ is a diffeomorphism onto $\mathrm{Ad}_B(\mathcal{D})$.*

Proof. Let $V \in \mathcal{P}$. Then it diagonalizes in an orthogonal basis: $B^\top V B = \Lambda$, with $B \in \mathrm{SO}(n)$ and $\Lambda \in \mathcal{D}$. Hence $\pi(B\Lambda^{1/2}) = V$, so $\pi|_{\mathcal{C}}$ is surjective. The fiber is

$$\pi|_{\mathcal{C}}^{-1}(V) = B\Lambda^{1/2}\,\mathrm{SO}(n)_\Lambda,$$

[3] One could also deduce involutivity by recalling that the curvature of the connection is $C \cdot [\mathfrak{s}, \mathfrak{s}]$, and, therefore, the abelian subalgebra $\mathfrak{d}$ is flat.

where the latter subgroup is the stabilizer of Λ for the adjoint action. The kernel of the differential of π at $B\Lambda^{1/2}$ is $B\Lambda^{1/2}\cdot\mathfrak{so}(n)$. The tangent space of $\mathcal{C}$ at $B\Lambda^{1/2}$ is $B\Lambda^{1/2}\cdot\mathrm{ad}_\Lambda^{-1}(\mathfrak{d})$. Because the adjoint orbit through Λ intersects $\mathcal{D}$ cleanly at Λ, their intersection – which is the kernel of the differential of $\pi|_{\mathcal{C}}$ at $B\Lambda^{1/2}$ – is $B\Lambda^{1/2}\cdot\mathfrak{so}(n)_\Lambda$. Therefore all values of $\pi|_{\mathcal{C}}$ are clean.

The tangent space to the leaf of $\mathcal{F}$ through $B\Lambda^{1/2}$ is $B\Lambda^{1/2}\cdot\mathfrak{d}$. Its intersection with $B\Lambda^{1/2}\cdot\mathfrak{so}(n)_\Lambda$ is trivial. Therefore, the restriction of π to $B\mathcal{D}$ is a local diffeomorphism over its image. That image is, by construction, $\mathrm{Ad}_B(\mathcal{D})$. To conclude that it is a diffeomorphism, one can either check that the map is bijective or argue that the manifolds involved are contractible. ∎

The base of the pencil (3.4) corresponds to inner products in the plane which have a unique eigenvalue. In arbitrary dimensions, we have analogous subsets. For each symmetric matrix, we can order its eigenvalues (with their multiplicity) in an increasing sequence. To each partition κ of $\{1,\dots,n\}$, there correspond a subset $\Theta^\kappa_{\mathfrak{s}}$; likewise, to each matrix with orthogonal columns we can order the norm of its columns in an increasing sequence. In that way we obtain partitions $\Theta_{\mathcal{D}}$, $\Theta_{\mathcal{P}}$ and $\Theta_{\mathcal{C}}$ of $\mathcal{D}$, $\mathcal{P}$ and $\mathcal{C}$, respectively.

Proposition 5.4. *The partitions $\Theta_{\mathcal{D}}, \Theta_{\mathcal{P}}$ and $\Theta_{\mathcal{C}}$ are stratifications of $\mathcal{D}$, $\mathcal{P}$ and $\mathcal{C}$, respectively, and they interact with the map $\pi|_{\mathcal{C}}\colon(\mathcal{C},\mathcal{F})\to\mathcal{P}$ as follows:*

(1) *The preimage of $\Theta^\kappa_{\mathcal{P}}$ is $\Theta^\kappa_{\mathcal{C}}$, and the restriction is a principal bundle:*

$$\pi|_{\Theta^\kappa_{\mathcal{C}}}:\Theta^\kappa_{\mathcal{C}}\to\Theta^\kappa_{\mathcal{P}}. \tag{5.2}$$

(2) *The foliation $\mathcal{F}=\mathrm{SO}(n)\mathcal{D}$ of $\mathcal{C}$ intersects the stratum $\Theta^\kappa_{\mathcal{C}}$ cleanly and induces there the foliation $\mathrm{SO}(n)\Theta^\kappa_{\mathcal{D}}$ of $\Theta^\kappa_{\mathcal{C}}$.*

(3) *The foliation $\mathrm{SO}(n)\Theta^\kappa_{\mathcal{D}}$ of $\Theta^\kappa_{\mathcal{C}}$ is projectable by the submersion $\pi|_{\Theta^\kappa_{\mathcal{C}}}$. Its image is the foliation $\mathrm{Ad}_{\mathrm{SO}(n)}(\Theta^\kappa_{\mathcal{D}})$ of $\Theta^\kappa_{\mathcal{P}}$.*

It is in this sense that $\pi\colon(\mathcal{C},\mathcal{F},\Theta_{\mathcal{C}})\to(\mathcal{P},\mathrm{Ad}_{\mathrm{SO}(n)}(\mathcal{D}),\Theta_{\mathcal{P}})$ is a desingularization of the stratified pencil.

Proof. The group $\mathrm{SO}(n)$ acts on $\mathfrak{s}$ by conjugation. As for any proper action, it produces a stratification of $\mathfrak{s}$ in orbit types (see Chapter 2 of [4]): two symmetric matrices are related if their isotropy subgroups are conjugated. It is well known that upon passing to connected components, the outcome is a (Whitney B) stratification of $\mathfrak{s}$. The stratification $\Theta_{\mathfrak{s}}$ is the result of possibly collecting some of the strata of the orbit type stratification belonging to the same subset of the orbit type partition; in any case, it is still a stratification for the partial order associated to the partitions κ of $\{1,\dots,n\}$. The stratification $\Theta_{\mathfrak{s}}$ – made of adjoint orbits – intersects the Cartan subalgebra $\mathfrak{d}$ cleanly, thus inducing a stratification $\Theta_{\mathfrak{d}}$. Each strata there is a face of the positive Weyl chamber of diagonal matrices with diagonal elements ordered increasingly; an open convex polytope in a vector subspace $\mathfrak{d}^\kappa$. The partition $\Theta_{\mathcal{P}}$ is obtained by intersecting $\Theta_{\mathfrak{s}}$ with the open subset of positive matrices, thus it is a stratification. The partition $\Theta_{\mathcal{D}}$ is also obtained upon intersection; it is a stratification because for instance $\mathcal{D}$ is an open subset of $\mathfrak{d}$. Finally, $\Theta_{\mathcal{C}}$ is the pullback of $\Theta_{\mathcal{D}}$ by the submersion q, and therefore it is a stratification as well.

Let $C\in\mathcal{C}$ with factorisation $C=B\Lambda$. Then $q(C)=B\Lambda^{-2}B^{\mathsf{T}}$, and therefore,

$$C\in\Theta^\kappa_{\mathcal{C}} \iff \Lambda\in\Theta^\kappa_{\mathcal{D}} \iff \Lambda^{-2}\in\Theta^\kappa_{\mathcal{D}} \iff q(C)\in\Theta^\kappa_{\mathcal{D}}.$$

The stratum $\Theta_k^{\mathcal{D}}$ is an open subset of the vector subspace $\mathfrak{d}^\kappa$ of all matrices whose stabilizer contains $\mathrm{SO}(n)_\kappa$. Because the fiber of $\pi|_{\mathcal{C}}$ through C is $C\,\mathrm{SO}(n)_\kappa$ and $\pi|_{\Theta_{\mathcal{C}}^\kappa}$ is saturated by fibers of $\pi|_{\mathcal{C}}$, it is a principal $\mathrm{SO}(n)_\kappa$-bundle. This proves (1).

The fibers of q are the orbits of the left $\mathrm{SO}(n)$-action. The restriction $q|_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}$ is the square map, which preserves the strata of $\Theta_{\mathcal{D}}$. Therefore the factorization of $\mathcal{C}$ is compatible with the stratification:

$$\Theta_{\mathcal{C}} = \mathrm{SO}(n)\Theta_{\mathcal{D}}.$$

Thus the intersection of the leaf of $\mathcal{F}$ though $C \in \Theta_{\mathcal{C}}^\kappa$ is $C\Theta_{\mathcal{D}}^\kappa$. At C, the respective tangent spaces are $C \cdot \mathfrak{d}$ and $C \cdot \mathfrak{d}^\kappa$. Therefore the intersection is clean and this proves (2).

By item (1) above and by item (2) in Proposition 5.3, the restriction of $\pi|_{\Theta_{\mathcal{C}}^\kappa}$ to the leaf $C \cdot \Theta_{\mathcal{D}}^\kappa \subset \Theta_{\mathcal{C}}^\kappa$ is a diffeomorphism over its image. Its image is $\mathrm{Ad}_B(\Theta_{\mathcal{D}}^\kappa)$ ($\pi|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}$ is the inverse of the square map); it is in fact the common image of all leaves through points of the fiber $C\,\mathrm{SO}(n)_\kappa$. ∎

We can now sharpen Proposition 4.2.

Proposition 5.5. *Let $H\phi$ be a Hessian metric on Ω such that $H\phi(\Omega)$ is contained in the stratum $\Theta_{\mathcal{P}}^\kappa$. Then its Christoffel symbols are symmetric if and only if $H\phi$ and $[H\phi]_k$ can be conjugated by an orthogonal matrix to a matrix in $\mathfrak{d}^\kappa$, $1 \le k \le n$.*

Proof. Because the image of $H\phi$ is contained in $\Theta_{\mathcal{P}}^\kappa$, its partial derivatives must be in the tangent space to the stratum: $[H\phi]_k \in T\Theta_{\mathcal{P}}^\kappa$. By item (2) in Proposition 4.2, there exists a special orthogonal matrix B which conjugates $H\phi$ and $[H\phi]_k$ to a diagonal one: $B^\mathsf{T}[H\phi]_k B \in \mathfrak{d}$. Therefore,

$$B^\mathsf{T}[H\phi]_k B \in \mathfrak{d} \cap T\Theta_{\mathcal{P}}^\kappa = \mathfrak{d}^\kappa. \qquad \blacksquare$$

6. Differential relations on the submanifold of matrices with orthogonal columns

We want to transfer property $\mathcal{J}$ for Hessian metrics into a differential condition for the orthogonal frame bundle at the submanifold of matrices with orthogonal columns.

Let $H\phi$ be a Hessian metric on $\Omega \subset \mathbb{R}^n$. To every curve $\gamma\colon (-\varepsilon, \varepsilon)$ based at $x \in \Omega$ we associate a curve in $\mathcal{P}$ based at $H\phi(x)$:

$$H\phi(\gamma) : (-\varepsilon, \varepsilon) \to \mathcal{P}, \quad t \mapsto H\phi(\gamma(t)).$$

Upon choosing an orthonormal frame for $H\phi(x)$, we can construct the horizontal lift of $H\phi(\gamma)$ based at the orthonormal frame.

Definition 6.1. A Hessian metric $H\phi$ in Ω has property $\mathcal{C}$ if for any point $x \in \Omega$ and any curve γ at $x \in \Omega$ there exists an orthonormal frame $C \in \mathcal{C}$ for $H\phi(x)$ such that the corresponding horizontal curve is tangent to $\mathcal{C}$ at C.

We now translate property $\mathcal{J}$ to the universal orthogonal frame bundle setting:

Proof of Theorem 1.5. We must show that a Hessian metric in $\Omega \subset \mathbb{R}^n$ has property $\mathscr{C}$ if and only if it has symmetric Christoffel symbols.

Property $\mathscr{C}$ is linear in the velocity of the curve at x. Thus it is enough to prove the equivalence for $\gamma(t) = x + te_k$, $1 \leq k \leq n$. Let us denote the horizontal lift at $A \in \mathscr{C}$ by $A(t)$. That A belongs to $\mathscr{C}$ means that $A^{\mathsf{T}}A = \Lambda \in \mathscr{D}$. By Proposition 5.1 (taken from [3]), the pullback of $\pi\colon (\mathrm{Gl}(n)^+, \nabla) \to \mathscr{P}$ by $H\phi$ is the orthonormal frame bundle of $H\phi$ with its Levi-Civita connection. Thus we have

$$0 = A'_{\bullet\star} + \Gamma^{\bullet}_{k\circ} A_{\circ\star}\ (= A' + \Gamma_k A).$$

The image by the differential of q of the vector of $A' = A'(0)$ is $A^{\mathsf{T}\prime}A + A^{\mathsf{T}}A'$. Therefore the Hessian metric satisfies property $\mathscr{C}$ at A if and only if

$$A^{\mathsf{T}}\Gamma_k^{\mathsf{T}}A + A^{\mathsf{T}}\Gamma_k A \in \mathfrak{d}. \tag{6.1}$$

We have $\Gamma_k = H\phi^{-1}[H\phi]_{,k}$, $H\phi^{-1} = AA^{\mathsf{T}}$, where the latter identity uses that A is an orthonormal frame for $H\phi$. Hence we may rewrite $\Gamma_k = AA^{\mathsf{T}}[H\phi]_{,k}$. Thus equation (6.1) is equivalent to

$$A^{\mathsf{T}}[H\phi]_{,k}AA^{\mathsf{T}}A + A^{\mathsf{T}}AA^{\mathsf{T}}[H\phi]_{,k}A = A^{\mathsf{T}}[H\phi]_k A\Lambda + \Lambda A^{\mathsf{T}}[H\phi]_{,k}A \in \mathfrak{d}.$$

Because Λ has non-zero positive entries if its anti-commutator with a matrix is diagonal, then the matrix must be diagonal. The conclusion is that property $\mathscr{C}$ is equivalent to

$$A^{\mathsf{T}}[H\phi]_{,k}A \in \mathfrak{d}, \quad A^{\mathsf{T}}A \in \mathscr{D}, \quad AA^{\mathsf{T}} = H\phi^{-1}.$$

By item (2) in Proposition 4.2, this is exactly the symmetry of the Christoffel matrices. ∎

We can verify that strictly convex functions with orthogonal characteristics satisfy property $\mathscr{C}$.

Lemma 6.2. *If $\mathcal{L}^2_C\phi \in \mathscr{D}$, $C \in \mathscr{C}$, then $H\phi$ satisfies property $\mathscr{C}$.*

Proof. By definition, $C^{\mathsf{T}}H\phi C = \Lambda$, $\Lambda \in \mathscr{D}$. Therefore, $C^{\mathsf{T}}[H\phi]_{,k}C \in \mathfrak{d}$. The matrix $C\Lambda^{-1/2}$ also belongs to $\mathscr{C}$ and it is an orthonormal frame for $H\phi$. Therefore,

$$(C\Lambda^{-1/2})^{\mathsf{T}}[H\phi]_{,k}C\Lambda^{-1/2} \in \mathfrak{d},$$

and thus property $\mathscr{C}$ holds. ∎

Next we analyze up to which extent Hessian metrics with property $\mathscr{C}$ are defined by functions with orthogonal characteristics. As we shall see in the examples in Section 7, for a metric with property $\mathscr{C}$ it may happen that horizontal lifts do not remain in $\mathscr{C}$. Theorem 6.5 will show that the geometric reason behind is that $\mathscr{C}$ exerts control on the horizontal lifts of $H\phi(\gamma)$ for all times, provided that $H\phi(\gamma)$ is contained in a single stratum of $\Theta_{\mathscr{P}}$. To exert control on lifts of curves that change strata, we need to constraint the lifts by definition:

Definition 6.3. A Hessian metric $H\phi$ on $\Omega \subset \mathbb{R}^n$ has property $\mathscr{C}\mathscr{K}$ if there exist a point $x \in \Omega$ and an orthonormal frame $C \in \mathscr{C}$ for $H\phi(x)$, such that for every curve in Ω based at x, its horizontal lift at C is contained in $\mathscr{C}$.

Theorem 6.4. *If a Hessian metric $H\phi$ on Ω has property $\mathcal{CK}$, then it solves $\mathcal{L}_C^2\phi \in \mathfrak{d}$, $C \in \mathcal{C}$.*

Furthermore, the following conditions are equivalent:

(1) *The image $H\phi(\Omega)$ is contained in the stratum $\Theta_{\mathcal{P}}^{\kappa}$.*

(2) *Property $\mathcal{CK}$ holds for all orthonormal frames in $\mathcal{C}$ over all points of $H\phi(\Omega)$.*

In either case, ϕ restricts to the leaves of the (*parallel*) *foliation determined by the* (*rotation of*) *the subspace $\mathfrak{d}^{\kappa}$ to a multiple of the standard quadratic form* (*up to an affine summand*).

Proof. Let x and C be a point and orthonormal frame for $H\phi(x)$ with respect which condition $\mathcal{CK}$ is defined. Because Ω is (path) connected for every point y, we have a curve γ starting at x and ending at y. By the hypotheses, the horizontal lift of $H\phi(\gamma)$ is a curve $A(t)$ contained in $\mathcal{C}$. Therefore, it is in $T\mathcal{C}$ and horizontal. By (2) in Proposition 5.2, $A(t)$ is contained in the leaf $C\mathcal{D}$ of $\mathcal{F}$: $A(t) = CD(t)$. Because $A(t)$ is a curve of orthogonal frames, $A(t)^{\mathsf{T}} H\phi(\gamma)A(t) = \mathrm{I}$. Therefore,

$$C^{\mathsf{T}} H\phi C = D(t)^{-2} \in \mathcal{D}.$$

Equivalently, if $C = B\Lambda$, then by (2) in Proposition 5.3, $\pi|_{\mathcal{C}}$ sends the leaf $C\mathcal{D}$ diffeomorphically onto $\mathrm{Ad}_B(\mathcal{D})$, which is where the image of $H\phi$ must be confined. The (local) splitting condition for $H\phi$ along the characteristics of C can be also argued as follows: that $A(t) \subset C\mathcal{D}$ implies that line fields at x spanned by each column of C are invariant by parallel transport. Thus the de Rham splitting theorem applies (and Hessian metrics restrict to Hessian metrics).

If $H\phi \subset \Theta_{\mathcal{P}}^{\kappa}$, then by item (1) in Proposition 5.4, the frame $C \in \mathcal{C}$ with respect to which $\mathcal{CK}$ is defined belongs to $\Theta_{\mathcal{C}}^{\kappa}$. By item (2) in the same proposition, all horizontal curves based at C must be contained in the leaf $C\Theta_{\mathcal{D}}^{\kappa}$ of the foliation of $\Theta_{\mathcal{C}}^{\kappa}$ induced by $\mathcal{F}$ upon clean intersection. The principal $\mathrm{SO}(n)_{\kappa}$-action takes these horizontal curves at C to horizontal curves in $\Theta_{\mathcal{C}}^{\kappa}$ based at any matrix in the fiber.

Conversely, let $C = B\Lambda$ and assume that the horizontal lifts at CB', $B' \in \mathrm{SO}(n)_{\Lambda}$, are contained in $\mathcal{C}$. Then they are inside the corresponding leaf of $\mathcal{F}$, and therefore,

$$H\phi(\Omega) \subset \bigcap_{B' \in \mathrm{SO}(n)_{\Lambda}} \mathrm{Ad}_{BB'}(\mathcal{D}) = \mathrm{Ad}_B\Big(\bigcap_{B' \in \mathrm{SO}(n)_{\Lambda}} \mathrm{Ad}_{B'}(\mathcal{D})\Big).$$

Because the exponential intertwines the adjoint action, the latter intersection can be understood in $\mathfrak{s}$. If $\Lambda \in \mathfrak{d}^{\kappa}$, we have

$$\mathfrak{d}^{\kappa} = \bigcap_{B' \in \mathrm{SO}(n)_{\Lambda}} \mathrm{Ad}_{B'}(\mathfrak{d}), \quad \mathrm{SO}(n)_{\Lambda} = \mathrm{SO}(n)_{\kappa}.$$

We used property $\mathcal{CK}$ with respect to an arbitrary point $x \in \Omega$. If we select the point whose image lies in the stratum of smallest dimension, then we conclude that $H\phi(\Omega)$ cannot leave that stratum.

If $H\phi \subset \Theta_{\mathcal{P}}^{\kappa}$, then at any point $x \in \Omega$ all orthonormal frames in $\mathrm{SO}(n)_{\kappa}$ are parallel. This means that the common eigendirections $\mathrm{Ad}_B(\mathfrak{d}^{\kappa})$ are parallel. Therefore the restriction of $H\phi$ to the foliation given by the parallel translates of $\mathrm{Ad}_B(\mathfrak{d}^{\kappa})$ in Ω is flat. Hence on each such (affine) subspace it is a quadratic form with equal eigenvalues. Therefore, in the local splitting of ϕ along orthogonal characteristics we shall have a multiple of the standard quadratic form along $\mathrm{Ad}_B(\mathfrak{d}^\kappa)$. ∎

The following result is more general than Theorem 1.2 in the introduction.

Theorem 6.5. *If a Hessian metric $H\phi$ on Ω has property $\mathcal{C}$ and $H\phi(\Omega)$ is contained in a stratum of $\Theta_{\mathcal{P}}$, then it has property $\mathcal{CK}$. In particular, ϕ has orthogonal characteristics.*

Proof. Let $H\phi(\Omega)$ be contained in $\Theta^\kappa_{\mathcal{P}}$. This stratum is foliated by $\mathrm{SO}(n)\Theta^\kappa_{\mathcal{D}}$, and we want to show that for each curve $\gamma(t)$ in Ω the derivative of $H\phi(\gamma(t))$ is tangent to this foliation. By property $\mathcal{C}$, for each t_0 there exists an orthogonal frame $C \in \mathcal{C}$ such that the horizontal curve $A(t)$ at C has derivative at zero tangent to $\mathcal{C}$:

$$\begin{aligned} A'(0) \in T\mathcal{C} \cap T\pi^{-1}(\Theta^\kappa_{\mathcal{P}}) &= C \cdot \mathrm{ad}^{-1}_\Lambda(\mathfrak{d}) \cap C \cdot (\mathfrak{so}(n) + \mathfrak{d}^\kappa) \\ &= C \cdot (\mathfrak{so}(n)_\Lambda + \mathfrak{d}^\kappa) = T\pi|^{-1}_{\mathcal{C}}(\Theta^\kappa_{\mathcal{P}}). \end{aligned}$$

Therefore, the intersection $\mathcal{C} \cap \pi|^{-1}_{\mathcal{C}}(\Theta^\kappa_{\mathcal{P}}) = \Theta^\kappa_{\mathcal{C}}$ is clean, and $A'(0)$ belongs to $\Theta^\kappa_{\mathcal{C}}$. Because the vector is horizontal, by item (2) in Proposition 5.4, it is tangent to the foliation $\mathrm{SO}(n)\Theta^\kappa_{\mathcal{D}}$. Thus, by item (3), the tangent vector of $H\phi(\gamma)$ at t_0 is tangent to the foliation $\mathrm{SO}(n)\Theta^\kappa_{\mathcal{D}}$. Because Ω is connected, this implies that $H\phi(\Omega)$ is contained in one of the leaves of $\mathrm{SO}(n)\Theta^\kappa_{\mathcal{D}}$. ∎

The following result is a more precise statement that Theorem 1.3 in the introduction.

Theorem 6.6. *Let $H\phi$ be a real analytic Hessian metric on $\Omega \subset \mathbb{R}^n$. Then ϕ has property $\mathcal{C}$ if and only if it has property $\mathcal{CK}$. In such case, $H\phi$ is the restriction to Ω of a product Hessian metric on a (rotated) cube.*

Proof. The stratification $\Theta_{\mathcal{P}}$ has a finite number of strata. Therefore there exists one stratum $\Theta^\kappa_{\mathcal{P}}$ whose pullback by $H\phi$ has non-empty interior $\Omega' \subset \Omega$. By Theorem 6.5 and Theorem 6.4,

$$H\phi(\Omega') \subset \mathrm{Ad}_B(\Theta^\kappa_{\mathcal{D}}), \quad B \in \mathrm{SO}(n).$$

In particular, $H\phi(\Omega)$ must be contained in the real analytic submanifold $\mathrm{Ad}_B(\mathcal{D})$[4]. By Proposition 5.4, the restriction

$$\pi|_{\mathcal{C}} : B\mathcal{D} \to \mathrm{Ad}_B(\mathcal{D})$$

is a diffeomorphism from a horizontal submanifold. Therefore all lifts of curves in Ω at $C \in \Theta^\kappa_{\mathcal{C}}$ are contained in $B\mathcal{D} \subset \mathcal{C}$, which is property $\mathcal{CK}$.

The image of Ω by the orthogonal projection onto a characteristic line is connected, and hence an interval. The restriction of ϕ to the foliation of Ω by affine lines parallel to the characteristic line is locally projectable. Because the interval has trivial topology local projections must agree on overlaps. ∎

[4]The real analytic closure of $\mathrm{Ad}_B(\Theta^\kappa_{\mathcal{D}})$ is the exponential of $\mathrm{Ad}_B(\mathfrak{d}^\kappa)$. This means that $H\phi(\Omega)$ can only intersect strata of dimension equal or less to that of $\Theta^\kappa_{\mathcal{P}}$; the equidimensional strata are those which under permutations go to open subsets of $\mathfrak{d}^\kappa$).

7. Bifurcation of orthogonal characteristics along quadratic forms

We shall construct in Proposition 7.1 a family of Hessian metrics with property $\mathcal{I}$ which do not have orthogonal characteristics, and, hence, by Theorem 6.4, which do not have property $\mathcal{CK}$. The family, despite being defined on domains which are not necessarily convex, will be also invariant under Legendre transform.

Firstly, we shall describe the domains Ω we are interested in. Let Ω^0 be an (open convex) *polytope*. By this we mean a domain defined as the points where a finite number of affine maps are strictly positive; the zero set of each such map is a *supporting hyperplane*. The closure of the polytope need not be compact. Let $\mathbb{R}^1_l \times \mathbb{R}^{n-1}_l$ be the result of applying an orthogonal transformation B_l, $1 \le l \le m$, to the splitting $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$. We consider the polytope $\Omega^1_l = I_l \times F_l$, where the factors are polytopes in $\mathbb{R}^1_l, \mathbb{R}^{n-1}_l$, respectively; we shall refer to $\mathbb{R}^1_l$ as the *primary characteristic* of Ω^1_l. We shall assume that $\mathbb{R}^1_l$ is oriented and we shall denote by p_l the infimum of the interval I_l (the interval may not be bounded from above). We shall refer to $H_l := p_l \times \mathbb{R}^{n-1}_l$ as the *primary supporting hyperplane*.

We shall assume that

- the polytopes $\Omega^0, \Omega^1_1, \dots, \Omega^1_m$ are disjoint and that Ω^1_i and Ω^1_j, $i \neq j$, have disjoint closure;
- the primary supporting hyperplane H_l for Ω^1_l is also a supporting hyperplane for Ω^0 and $\partial\Omega^1_l \cap H_l \subset \partial\Omega^0 \cap H_l$.

We define Ω to be

$$\Omega = \Omega^0 \bigcup (\Omega^1_1 \cup p_1 \times F_1) \bigcup \cdots \bigcup (\Omega^1_m \cup p_m \times F_m). \tag{7.1}$$

We refer to Ω as in (7.1) as a *polytope with* 1*-handles*.

Secondly, we shall introduce appropriate strictly convex functions on the polytope with 1-handles. Let ϕ_0 be a multiple of the standard quadratic form on $\mathbb{R}^n$:

$$\phi_0(x) = k(x_1^2 + \cdots + x_n^2).$$

Let $y = (y_1, \dots, y_n)$ be the coordinates which correspond to the image by B_l of the canonical basis $e_1, \dots, e_n$ and let $q_l(y) = q_l(y_2, \dots, y_l) = k(y_2^2 + \cdots + y_n^2)$. Then we have:

$$\phi_0(y) = k(y_1^2 + \cdots + y_n^2) = k y_1^2 + q_l(y_2, \dots, y_l). \tag{7.2}$$

Proposition 7.1. *Let ϕ be the function defined on the polytope with* 1*-handles Ω as follows:*

$$\phi|_{\Omega^0 \cup p_1 \times F_1 \cup \cdots \cup p_m \times F_m} = \phi_0, \quad \phi|_{\Omega^1_l} = \phi_l + q_l,$$

where $\phi_l(y) = \phi_l(y_1)$ a strictly convex smooth function on I_l tangent at p_l to $k y_1^2$ at infinite order. Then it has the following properties:

(1) *It is a smooth and strictly convex function on Ω.*

(2) *It has property $\mathcal{I}$.*

(3) *The image $H\phi(\Omega) \subset \mathcal{P}$ is contained in the union of the closed stratum and the two open strata of lowest dimension of $\Theta_{\mathcal{P}}$.*

(4) *If there are two* 1*-handles on which* ϕ_l *is not a quadratic form and the primary characteristics are neither equal not perpendicular, then* $H\phi$ *does not have property* $\mathcal{CK}$.

Proof. That ϕ is smooth and strictly convex is a consequence of (7.2) and of the definition of ϕ_l.

The restriction of ϕ to Ω^0 has property $\mathcal{I}$; the restriction to each Ω^1_l has orthogonal characteristics given by the columns of B_l. Therefore ϕ has property $\mathcal{I}$ on the closure of the union, which is Ω.

By construction, $H\phi$ sends Ω^0 to the closed stratum of $\Theta_{\mathcal{P}}$, and thus so $\overline{\Omega^0}$; if ϕ_l is not a quadratic form, then $H\phi$ sends an open subset of Ω^1_l to the strata positive matrices with two eigenvalues so that one is simple:

$$H\phi(\overline{\Omega^0}) \subset \bigcap_{B\in \mathrm{SO}(n)} \mathrm{Ad}_B(\mathcal{D}), \quad H\phi(\Omega^1_l) \cap \mathrm{Ad}_{B_l}\Big(\bigcap_{B'\in S(\mathrm{O}(1)\times \mathrm{O}(n-1))} \mathrm{Ad}_{B'}(\mathcal{D})\Big) \neq \emptyset.$$

By Theorem 6.4, if $H\phi$ has property $\mathcal{CK}$, then ϕ has orthogonal characteristics for some $C \in \mathcal{C}$ (or $B \in \mathrm{SO}(n)$). On a 1-handle Ω^1_l with ϕ_l different from a quadratic form, ϕ has orthogonal characteristics exactly for all $B_l \mathrm{SO}(n)_\kappa \mathcal{D}$. The 1-handles Ω^1_i and Ω^1_j have primary characteristics which are neither equal nor orthogonal if and only if $B_i \mathrm{SO}(n)_\kappa \cap B_j \mathrm{SO}(n)_\kappa = \emptyset$. ∎

Proposition 7.2. *Let* Ω *be a polytope with* 1*-handles and let* $\phi \in C^\infty(\Omega)$ *be as in Proposition* 7.1. *Then the following holds*:

(1) *The Legendre map* $d\phi$ *on* Ω *is a diffeomorphism, its image* Ω^* *is a polytope with* 1*-handles, and* ${\Omega^1_l}^*$ *and* Ω^1_l *have the same primary characteristic.*

(2) ϕ^* *is a function as in Proposition* 7.1.

Proof. Because ϕ is strictly convex, $d\phi\colon \Omega \to \mathbb{R}^n$ is a local diffeomorphism. Because $\phi|_{\Omega^0}$ is a quadratic form, $d\phi(\Omega^0)$ is another polytope. The restriction $\phi|_{\Omega^1_l = I_l\times F_l}$ decomposes as a sum of strictly convex functions $\phi_l + q_l$. The subset $d\phi(\Omega^1_l)$ is another 1-handle because I_l is 1-dimensional, F_l is a polytope, and q_l is a quadratic form; furthermore, because the Legendre transform commutes with orthogonal transformations, $d\phi(\Omega^1_l) = I^*_l \times F^*_l$, where the product decomposition is also with respect to $\mathbb{R}^1_l \times \mathbb{R}^{n-1}_l$. The condition on the non-overlap of the closures of the 1-handles can be restated as follows: if two 1-handles have common primary supporting hyperplane, then their polytopes there have non-intersecting closure. This implies that if we prolong each $I_l \subset \mathbb{R}^1_l$ across p_l to a larger interval $\tilde{I}_l$ so that $\tilde{I}_l \times F_l \subset \Omega^0 \cup p_l \times F_l \cup \Omega^1_l$, then

$$d\phi(\Omega^0) \cap d\phi(p_l \times F_l \cup \Omega^1_l) = \emptyset,$$
$$d\phi(\tilde{I}_i \times F_i) \cap d\phi(\tilde{I}_j \times F_j) \cap (\mathbb{R}^n \backslash d\phi(\Omega^0) = \emptyset, \quad i \neq j.$$

Therefore, $d\phi\colon \Omega \to d\Omega$ is a bijection and thus a diffeomorphism onto another polytope with 1-handles.

Because the Legendre transform of a multiple of the standard quadratic form is a multiple of the standard quadratic form, it follows that ϕ^* belongs to the class of functions described in Proposition 7.1. ∎

8. An application to Poisson geometry

We shall show that property $\mathcal{J}$ for strictly convex functions is equivalent to the Poisson commuting equation for Poisson structures related to Kähler forms on toric varieties. We shall use

(a) our classification of real analytic inversible Hessian metrics to deduce a factorization result;

(b) the family of strictly convex functions with property $\mathcal{J}$ introduced in Section 7 to produce pencils of Poisson structures on regions of projective varieties which interpolate from a Kähler structure to a Poisson structure with a finite number of Kähler leaves.

Definition 8.1 (Section 4 in [2]). A Poisson structure Π on a toric variety $(X, \mathbb{T})$ is

(1) *toric* if the bivector field Π is $\mathbb{T}$-invariant, of type $(1, 1)$ and positive[5], and if the symplectic leaves of Π equal the finitely many orbits of the torus action;

(2) *totally real* if the orbits of the (maximal) compact torus T are coisotropic submanifolds.

Remark 8.2. Totally real toric Poisson structures are good candidates to be limits of Hamiltonian Kähler forms: for such a form, its inverse Poisson bivector is T-invariant, of type $(1, 1)$ and positive; there is a unique symplectic leaf of which the T-orbits are Lagrangian submanifolds. Thus it is natural to look for converging sequences of such bivectors so that in the limit the unique symplectic leaf breaks into the finitely many orbits, and the T-symmetry is enlarged to $\mathbb{T}$-symmetry. One possible source would be a totally real toric Poisson structure which Poisson commutes with (the inverse of) a Hamiltonian Kähler form. In such case, the convex combination of the bivectors would be a smooth family of (inverses of) Hamiltonian Kähler forms converging to the totally real toric Poisson structure.

On a toric variety, a $\mathbb{T}$-invariant Poisson structure has a simple infinitesimal description: its restriction to the open dense orbit (which, upon fixing a point, is identified with $\mathbb{T}$) followed by the logarithm map, defines a constant Poisson structure in the Lie algebra of $\mathbb{T}$. The infinitesimal counterpart of a toric Poisson structure is a Hermitian inner product. If it is totally real, it corresponds to an inner product on $i\mathfrak{t}$, where $\mathfrak{t}$ denotes the Lie algebra of T. In such case, we say that $e_1, \dots, e_n \in i\mathfrak{t}$ is an *adapted Darboux basis* if the inner product becomes standard; equivalently, $e_1, \dots, e_n, ie_1, \dots, ie_n$ is a Darboux basis for the inverse constant symplectic structure.

On a toric variety, a Hamiltonian Kähler form can be described by (Legendre dual) strictly convex functions: a *Kähler potential* in logarithmic coordinates and a *symplectic potential* in momentum map coordinates [5].

The Poisson commuting equation for a totally real Poisson structure and a Hamiltonian Kähler form corresponds – in appropriate coordinates – to property $\mathcal{J}$:

[5] The (real) quadratic form $\xi \mapsto \xi(J\Pi^{\#}(J^{*}\xi))$, $\xi \in T_x^{*}X$, is semi-positive and only vanishes in the kernel of $\Pi^{\#}$.

Theorem 8.3. *Let $(X, \mathbb{T})$ be a toric variety endowed with a totally real toric Poisson structure Π and a Kähler structure σ for which the action of T is Hamiltonian. Let P be the inverse Poisson structure of σ. Then the following statements are equivalent*:

(1) Π *and* P *Poisson commute*: $[\Pi, P] = 0$.

(2) *In an adapted Darboux basis, the Kähler potential ϕ has property $\mathcal{J}$.*

(3) *In an adapted Darboux basis, the symplectic potential ϕ^* has property $\mathcal{J}$.*

Proof. We regard the equation $[\Pi, P] = 0$ as the defining equation for degree 2-cocycles in the Poisson cohomology of Π: $d_\Pi P = 0$. In logarithmic coordinates, $\exp^* \Pi$ has an inverse which is a (constant) symplectic structure Ξ on $\mathfrak{t} \oplus i\mathfrak{t}$. Therefore,

$$\Xi^\# : (\mathfrak{X}^\bullet, d_{\exp^* \Pi}) \to (\Omega^\bullet, d)$$

is an isomorphism of chain complexes (see e.g. Proposition 6.12 in [6]). Hence

$$[\exp^* \Pi, \exp^* P] = 0 \iff d\omega = 0, \quad \omega(X, Y) := \exp^* P(\Xi^\# X, \Xi^\# Y).$$

Let $e_1, \dots, e_n, ie_1, \dots, ie_n$ be an adapted Darboux basis for the totally real toric Poisson structure. In the fixed coordinates and associated frames of the complexified tangent and cotangent bundles, the matrices of Π and Ξ are

$$\Pi^\# = \frac{2}{i}\begin{pmatrix} 0 & \mathrm{I}_n \\ -\mathrm{I}_n & 0 \end{pmatrix} \quad \text{and} \quad \Xi^\# = \frac{i}{2}\begin{pmatrix} 0 & -\mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix}.$$

If we let

$$\frac{2}{i}\begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}$$

denote the matrix of $\exp^* P^\#$, then

$$\omega^\# = -\Xi^\# \exp^* P^\# \Xi^\# = -\frac{i}{2}\begin{pmatrix} 0 & -\mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix} \frac{2}{i}\begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix} \frac{i}{2}\begin{pmatrix} 0 & -\mathrm{I}_n \\ \mathrm{I}_n & 0 \end{pmatrix} = \frac{i}{2}\begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix}.$$

Hence

$$\omega = \omega(x) = \frac{-i}{2}\sum_{j,k} g_{jk}(x)\, dz_j \wedge d\bar{z}_k,$$

Its exterior derivative is

$$\begin{aligned} d\omega &= -\frac{i}{2}\sum_{i,j=1}^n dg_{ij} \wedge dz_i \wedge d\bar{z}_j = -\frac{i}{2}\sum_{i,j=1}^n \sum_{k=1}^n \Big(\frac{\partial g_{ij}}{\partial z_k}\, dz_k + \frac{\partial g_{ij}}{\partial \bar{z}_k}\, d\bar{z}_k\Big) \wedge dz_i \wedge d\bar{z}_j \\ &= -\frac{i}{2}\Big(\sum_{i,j,k=1}^n \Big(\frac{\partial g_{jk}}{\partial z_i} - \frac{\partial g_{ik}}{\partial z_j}\Big) \wedge dz_i \wedge dz_j \wedge d\bar{z}_k \\ &\quad + \sum_{i,j,k=1}^n \Big(\frac{\partial g_{ij}}{\partial \bar{z}_k} - \frac{\partial g_{ik}}{\partial \bar{z}_j}\Big) \wedge dz_i \wedge d\bar{z}_j \wedge d\bar{z}_k\Big). \end{aligned}$$

Because the entries of g only depend on x, we have

$$\frac{\partial g_{jk}}{\partial z_i} - \frac{\partial g_{ik}}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_j}\Big) \quad \text{and} \quad \frac{\partial g_{ij}}{\partial \bar{z}_k} - \frac{\partial g_{ik}}{\partial \bar{z}_j} = \frac{1}{2}\Big(\frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_j}\Big).$$

The matrix of $\exp^* \sigma$ is

$$\frac{i}{2}\begin{pmatrix} 0 & g^{-1} \\ -g^{-1} & 0 \end{pmatrix},$$

and g^{-1} is the Hessian of the Kähler potential ϕ. In particular, g and its inverse are symmetric matrices. Renaming the set of indices in the first summand and using the symmetry of g, we obtain

$$d\omega = 0 \iff \frac{\partial g_{ij}}{\partial x_k} - \frac{\partial g_{ik}}{\partial x_j}, \quad 1 \leq i, j, k \leq n.$$

This is exactly property $\mathcal{J}$ for the Hessian metric $g^{-1} = H\phi$.

The symplectic potential of σ is the Legendre transform of ϕ. By Proposition 4.4, ϕ (in $\mathbb{R}^n$) has property $\mathcal{J}$ if and only if ϕ^* (in $d\phi(\mathbb{R}^n)$) has property $\mathcal{J}$. ■

Theorem 8.4. *Let $(X, \mathbb{T})$ be a projective toric variety endowed with a toric Poisson structure Π which Poisson commutes with a real analytic Kähler structure σ for which the action of T is Hamiltonian. Then $(X, \mathbb{T})$ is a Cartesian product of projective lines, and both Π and σ factorize.*

Proof. Because σ is real analytic, the Kähler potential ϕ is real analytic; the Legendre transform preserves analytic (strictly convex) functions. Therefore, by Theorem 8.3, the symplectic potential ϕ^* has property $\mathcal{J}$. By Theorem 6.6, ϕ^* is defined in a Cartesian product of intervals $I_1 \times \cdots \times I_n$ (we may dispense with the rotation by changing accordingly the adapted Darboux basis). One must have the equality $d\phi(\mathbb{R}^n) = I_1 \times \cdots \times I_n$ because otherwise by repeating the Legendre transform we would get a domain for the original Kähler potential strictly containing $\mathbb{R}^n$. Thus the interior of the moment polytope Δ is a Cartesian product of intervals. A product of intervals has a property invariant under affine transformations: it is limited by pairs of parallel hyperplanes. Because Δ is a Delzant polytope, there is an affine transformation that takes the integral lattice of $\mathfrak{t}^*$ to $\mathbb{Z}^n$, a vertex of Δ to the origin, and the facets containing this vertex to the coordinate hyperplanes. Because Δ must be still described by parallel hyperplanes, is it actually a Cartesian product of intervals in this integral affine coordinates of $\mathfrak{t}^*$. The fan of the Delzant polytope determines the toric variety $(X, \mathbb{T})$. The fan of a cube in $(\mathbb{R}^n, \mathbb{Z}^n)$ corresponds to $\mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$.

To show that the Kähler form σ also splits as a sum of Kähler forms on each projective line, we use toric charts for $(X, \mathbb{T})$. For that we observe that the linear part of the above affine transformation must be a permutation followed by a (signed) re-scaling of each Euclidean direction. Therefore, if we dispense the affine transformation, we deduce that in the fixed compatible Darboux basis, the subset $ie_1, \ldots, ie_n$ is – up to re-scaling of its members – an integral basis of $\mathfrak{t}$. Let us re-scale to a basis $\varepsilon_1, \ldots, \varepsilon_n, i\varepsilon_1, \cdots, i\varepsilon_n$ so that the second block is an integral basis of $i\mathfrak{t}$. In the corresponding coordinates, the Kähler potential of σ is still a sum of strictly convex functions on each coordinate, and therefore the Legendre diffeomorphism still sends $i\mathfrak{t} \cong \mathbb{R}^n$ to a cube. Let v be the vertex of its closure whose coordinates are smaller than those of the others. The basis of inner pointing integral vectors normal to the facets containing v is exactly $i\varepsilon_1, \dots, i\varepsilon_n$. Therefore, for the standard toric chart associated to v (see, e.g., Section 5 of Chapter 2 in [1]),

$$(\mathbb{C}^*)^n \circlearrowright \mathbb{C}^n \subset \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1, \quad (0, \dots 0) = ([0:1], \dots, [0:1]),$$

the identification[6] of $\mathbb{T}$ with $(\mathbb{C}^*)^n$ comes from the Lie algebra identification which sends $i\varepsilon_1, \dots, i\varepsilon_n$ to the canonical basis of $\mathbb{R}^n \subset \mathbb{C}^n$. In other words, upon having identified $\mathbb{T}$ with the open orbit of X, the product structure induced by $X \cong \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ on $\mathbb{T}$ is exactly the factorisation of the torus coming from the (complex) basis $e_1, \cdots, e_n$ of its Lie algebra. The factorisation of X decomposes $\sigma = \sigma_1 + \cdots + \sigma_n$, $\sigma_k \in \Omega^2(X)$. Each σ_k is basic for the k-th projection

$$X \cong \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1 \to \mathbb{C}P^1,$$

because it has that property in the open dense subset $\mathbb{T}$. The compatibility of Π with the product structure is also immediate. Therefore,

$$(X, \mathbb{T}, \Pi, \sigma) \cong (\mathbb{C}P^1, \mathbb{C}^*, \Pi_1, \sigma_1) \times \cdots \times (\mathbb{C}P^1, \mathbb{C}^*, \Pi_n, \sigma_n),$$

where symplectic forms are invariant under the action of T. ∎

Let Ω be a polytope with 1-handles. Its *outer boundary* will be the subset of the boundary lying in supporting hyperplanes for the 1-handles which are parallel to the primary ones but not equal to them; its *inner boundary* will be the subset of the boundary in supporting hyperplanes of the polytope which are not primary supporting hyperplanes of some 1-handle.

Theorem 8.5. *Let $(X, \mathbb{T})$ be a toric variety with fan given by the polytope $\Delta \subset (\mathbb{R}^n, \mathbb{Z}^n)$. Let Ω be a polytope with 1-handles such that the intersection $\partial\Omega \cap \partial\Delta$ is contained in the union of the outer boundary of Ω and an orthogonal subset of supporting hyperplanes of the inner boundary of Ω.*

Then there exist an open subset $X_\Omega \subset X$ invariant under the action of T and a Kähler form $\sigma \in \Omega^2(X_\Omega)$ with the following properties:

(1) *The action of T on (X_Ω, σ) is Hamiltonian with momentum map the union of Ω with the interior on each face of Δ of the outer and inner boundaries of Ω.*
(2) *The Poisson structure which corresponds to σ Poisson commutes with any toric Poisson structure for which the canonical basis of $\mathbb{R}^n$ is an adapted Darboux basis up to scaling.*

Proof. We start with a function $\phi \in C^\infty(\Omega)$ as in Proposition 7.1, on which we shall impose natural boundary conditions[7]. Let Ω^1_l be a 1-handle whose supporting hyperplane

[6] Though we do not need it here, the toric chart could be chosen compatible with the monoid structure, so that $(1, \dots, 1)$ corresponds to the fixed point in the open orbit.

[7] The functions will satisfy well-known boundary conditions to produce Kähler metrics (see, e.g., Chapter 2 in [1]). These metrics/complex structures are constructed fixing the symplectic structure. Because we are interested in keeping fixed the complex structure, we are going to be very explicit with the computation of the Kähler potentials and corresponding symplectic forms.

in the outer boundary of Ω intersects $\partial\Delta$. Let α_l be the unique integral affine map which vanishes in the supporting hyperplane and it is positive on Δ. We shall assume that ϕ_l equals $\frac{1}{2}\alpha_l(\log(\alpha_l)-1)$ near the end of I_l opposite to p_l. This is always possible because the existing constraint on ϕ_l is near p_l.

The region $X_\Omega \subset X$ is the result of adding certain points to $\exp(\Omega^* \oplus i\mathbb{R}^n) \subset X$. By item (1) in Proposition 7.2, the primary characteristics of ${\Omega_l^1}^*$ and Ω_l^1 are the same, and the orientation is also preserved. Because Δ determines a fan, the primary characteristic $\mathbb{R}_l^1 \subset \mathbb{R}^n \cong i\mathfrak{t}$ determines a 1-parameter subgroup $\mathbb{C}_l^* \subset \mathbb{T}$ together with an isomorphism $\mathbb{C}_l^* \cong \mathbb{C}^*$ (the Lie algebra is trivialized by a positive integral vector in $\mathbb{R}_l^1$). Because the derivative of ϕ_l near the boundary point opposite to p_l goes to infinity, $I_l^* \subset \mathbb{R}_l^1$ is a semi-infinite interval in the positive half line; in particular, it is a semigroup; let $\mathbb{D}_l^o \subset \mathbb{C}_l^*$ be the semigroup $\exp(I_l^* \oplus i\mathbb{R}_l^1)$. This semigroup acts (freely) on $\exp({\Omega_l^1}^* \oplus i\mathbb{R}^n)$. If we let $\mathbb{T}_l \subset \mathbb{T}$ be the subtorus which exponentiates the complexification of $\mathbb{R}_l^{n-1}$, then we can factor

$$\exp({\Omega_l^1}^* \oplus i\mathbb{R}^n) = \mathbb{D}_l^o \times \exp(F_l \oplus i\mathbb{R}_l^{n-1}) \subset \mathbb{T} = \mathbb{C}_l^* \times \mathbb{T}_l. \tag{8.1}$$

We define $X_\Omega \subset X$ to be the union of $\exp(\Omega^* \oplus i\mathbb{R}^n)$ with the closure of every $\mathbb{D}_l^o$-orbit in $\exp({\Omega_l^1}^* \oplus i\mathbb{R}^n)$, for every 1-handle whose supporting hyperplane in the outer boundary of Ω intersects $\partial\Delta$ (for the moment we assume that no inner boundary components are in $\partial\Delta$).

The function ϕ^* defines a Kähler form $\sigma = i\partial\bar{\partial}\log(\phi^*)$ on $\exp(\Omega^* \oplus i\mathbb{R}^n)$ for which the action of T is Hamiltonian. We want to argue that σ extends to a Kähler form on X_Ω.

Firstly, we show how upon adding the orbit closures to $\exp({\Omega_l^1}^* \oplus i\mathbb{R}^n)$ (and not in $\exp(\Omega^* \oplus i\mathbb{R}^n)$) we get an open subset $X_l \subset X$ which is T-invariant and to which the product structure in (8.1) extends. For that we use the toric atlas (as a monoid) determined by the polytope Δ (see Section 5 in Chapter 2 of [1]): to each vertex $v \in \Delta$, there correspond a toric chart which identifies the union of orbits of X which correspond to the star of v with the standard affine toric variety: $(\mathbb{C}^n, (\mathbb{C}^*)^n)$. The standard integral basis of the Lie algebra of $(\mathbb{S}^1)^n$ comes from the integral linear forms $\nu_{l_1}, \dots, \nu_{l_n}$ associated to the affine forms α_{l_j}; in particular, this describes how for each toric chart $\mathfrak{t} \oplus i\mathfrak{t}$ – for which we already have picked a basis – is identified with the Lie algebra of the standard complex torus $\mathbb{R}^n \oplus i\mathbb{R}^n$. The point in the open orbit of X which determines the monoid structure goes to the unit $(1, \dots, 1) \in \mathbb{C}^n$.

Let us fix a toric chart of a vertex v which belongs to the supporting hyperplane of Ω_l^1 and Δ. Under the identification of $\mathbb{T}$ with the standard torus $(\mathbb{C}^*)^n$, we can assume that the (trivialized) subgroup $\mathbb{C}_l^* \cong \mathbb{C}^*$ maps to the first factor of the standard torus so that on trivializations the isomorphism is given by the inversion. Thus $\mathbb{D}_l^o$ maps to a semigroup $\mathbb{D}_l^i \subset \mathbb{C}^*$ contained in the unit disk. Let $W_l \subset \mathbb{C}^n$ be the image in the toric chart of the second factor $\exp(F_l \oplus i\mathbb{R}_l^{n-1})$. Then the image of $\exp({\Omega_l^1}^* \oplus i\mathbb{R}^n)$ is

$$(zw_1, \dots, w_n), \quad z \in \mathbb{D}_l^o, \quad w = (w_1, \dots, w_n) \in W_l.$$

Therefore the image of X_l is also completely contained in the toric chart and equals

$$(zw_1, \dots, w_n), \quad z \in \mathbb{D}_l^o \cup \{0\}, \quad (w_1, \dots, w_n) \in W_l.$$

Because W_l is a codimension 2 submanifold which intersects each complex line parallel to the z_1-axis transversely in at most one point, we deduce that X_l is an open subset which extends the product structure. By construction, it is also T-invariant.

To show that σ extends to a Kähler form on X_l, we shall work with its inverse Poisson structure P. The decomposition

$$\phi^*|_{\Omega_l^{1*}}(y) = \phi_l^*(y_1) + q_l^*(y_2, \dots, y_l)$$

implies that on the image of the exponential map of the 1-handle, the bivector P decomposes as $P_l + P_l'$, where each summand is a field of bivectors tangent to one of the foliations in the product decomposition (8.1). The second field of bivectors is easier to describe: in the Lie algebra, the foliation is given by translates of $F_l \subset \mathbb{R}_l^{n-1}$. On each such leaf, the Kähler potential for the corresponding Kähler form is the quadratic form q_l. Therefore P_l' corresponds to a constant bivector on $\Omega^{1^*} \times i\mathbb{R}^n$. The exponentiation of a constant bivector to the (abelian) Lie group has an alternative description: it is the field of bivectors obtained by replacing each vector in the decomposition in $\wedge^2(\mathfrak{t} \oplus i\,\mathfrak{t})$ by its corresponding infinitesimal vector field for the action by (left) multiplication. Because the action of $\mathbb{T}$ on itself extends to an action on X, it follows that P_l' is the restriction of a (Poisson) structure on X. To describe P_l on each semigroup orbit, we may assume that $\phi_l(y_1)$ equals $-\frac{1}{2}\alpha_l(\log(\alpha_l - 1))$ everywhere in I_l. The Legendre transform of $-\frac{1}{2}(-r)(\log(-r) - 1)$ for $r < 0$ is $\frac{1}{2}\exp^{-2r}$. Using that the Legendre transform commutes with orthogonal transformations and its behavior under translation and scaling, we obtain

$$\phi_l^*(y_1) = \frac{1}{2}\exp^{-2y_1/|v_l|} - y_1 d_l,$$

where d_l is the distance of the boundary point of I_l different from p_l. Since we are interested in Kähler forms/bivectors we may dispense with the linear summand. Under the semigroup identification $\log\colon \mathbb{D}_l^o \to I_l^* \oplus i\mathbb{R}$, the potential pullbacks to $1/(2z\bar{z})$. Under the identification $\mathbb{D}_l^o \to \mathbb{D}_l^i$, it maps to $\frac{1}{2}z\bar{z}$. Under the action on the standard toric chart, it maps to $\frac{1}{2a_1^2}z_1\bar{z}_1$. Hence the bivector P_l there equals $\frac{2a_1^2}{i}\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial \bar{z}_1}$ near $z_1 = 0$, which extends to X_l. Both P_l and P_l' are non-degenerate in the added points, and thus σ extends to a Kähler form.

For boundary components in the inner boundary, we change coordinates by a rotation so that all supporting hyperplanes involved are coordinate hyperplanes. Then we impose the same boundary conditions as above on the corresponding summands of the multiple of the standard quadratic form in these coordinates. We may have supporting hyperplanes with non-empty intersection, say k of them. This means that we shall have to work with the corresponding coordinates and hence with a splitting into a vector subspace of dimension k and its orthogonal complement. We shall work on a toric chart associated to a vertex in the intersection of the supporting hyperplanes. There, the foliation corresponding to the vector subspace will have leaves given by the action of $(\mathbb{C}^*)^k$ on the first k coordinates on an appropriate slice. Hence by adding the closure of (semigroup) orbits, we shall obtain an open subset. The computation of the Kähler potential for the inverse symplectic form of P_l on such leaves is analogous.

The computation of the image of the momentum map is straightforward. ■

The proof of Proposition 1.9 in the introduction is a minor variation of the following.

Example 8.6 (Attaching a toric 1-handle to the standard commuting pair). Let Δ be the standard n-simplex in $\mathbb{R}^n$. Let Ω^0 be the truncation of the (open) cube of side $(0, 1/n)$ by the hyperplane $x_1 + \cdots + x_n = 2/3$. Let Ω^1 be the 1-handle with the primary characteristic spanned by $(1, \dots, 1)$, and so that its primary supporting hyperplane is $x_1 + \cdots + x_n = 2/3$, the parallel one is $x_1 + \cdots + x_n = 1$, and the $(n-1)$-dimensional polytope is the (translation of) the intersection of the cube and the primary supporting hyperplane.

We let Ω be the polytope with 1-handles determined by Ω^0 and Ω^1 above. It satisfies the hypotheses of Theorem 8.5. By going through its proof, we check that:

- In the toric chart associated to the origin, σ (near the origin) will be the standard (constant) Kähler form $\frac{i}{2}\sum_j dz_j \wedge d\bar{z}_j$; the open cube is a (punctured) polydisk which is appropriately truncated.
- Near the truncation hypersurface W, the Kähler form is
$$\frac{i}{2}\sum_j \frac{1}{z_j \bar{z}_j} dz_j \wedge d\bar{z}_j.$$
- Attaching the 1-handle amounts to the following: the truncating hypersurface W is stable under the diagonal action of $\mathbb{S}^1$. Then each such orbit is being 'capped' by a (holomorphic) disk which is Kähler for σ; the disk is nothing but (a part of) the projective line determined by the orbit, its center being in the hyperplane at infinity $\mathbb{C}P^n = \mathbb{C}^n \cup \mathbb{C}P^{n-1}$; this is done for the whole $F_1 \times (\mathbb{S}^1)^{n-1}$-family.
- The inverse of Kähler form σ is a Poisson bivector field P which Poisson commutes with the totally real toric Poisson bivector field Π which in the previous toric chart is
$$\frac{2}{i}\sum_j z_j \bar{z}_j \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial \bar{z}_j}.$$

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